Relativization barriers and the universality of poly-time Turing equivalence

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1 Introduction

1.1 Universal resource bounded equivalence relations from computability

In this paper, we study the global complexity of resource bounded reducibilities from computability theory. If $\mathcal{C}$ is a subset of the Turing reductions such as the poly-time Turing reductions, and $x, y \subseteq 2^{<\omega}$ are languages, then we write $x \leq_T^{\mathcal{C}} y$ if x is Turing reducible to y via a Turing reduction in $\mathcal{C}$. Similarly, if $\mathcal{C}$ is a subset of the many-one reductions, then we analogously define $x \leq_m^{\mathcal{C}} y$ if x is many-one reducible to y via a many-one reduction from $\mathcal{C}$. Now these relations are not transitive in general. However, we will be particularly interested in the special case where we fix a function $g \colon \mathbb{N} \to \mathbb{N}$ which is strictly increasing, time constructible, and $g(n) \geq n^2$, and let $\mathcal{C}$ be the reductions computable in $O(g^k)$ time or $O(g^k)$ space. That is, $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \mathbf{TIME}(O(g(n)^k))$ or $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \mathbf{SPACE}(O(g(n)^k))$. In this case, since there is only a polynomial amount of overhead required to simulate the composition of two reductions, it is easy to see that the associated reducibilities are transitive and symmetrize to equivalence relations which we study below.

Previous investigations of the global properties of resource bounded reducibilities in computability have focused mainly on the theory of these structures [1, 6, 9, 17]. We take a different approach, using the framework of Borel reducibility between Borel equivalence relations. A *Borel equivalence relation* E on a standard Borel space X is an equivalence relation that is Borel subset of $X \times X$. If E and F are both Borel equivalence relations on X and Y, then E is *Borel reducible* to F, noted $E \leq_B F$, if there is a Borel function $f \colon X \to Y$ such that for all $x, y \in X$, we have $x \mathrel{E} y \iff f(x) \mathrel{F} f(y)$.

Such a function induces an injection $\hat{f} \colon X/E \to Y/F$, and we can view Borel reducibility as comparing the difficult of classifying E and F by invariants, where if $E \leq_B F$, then any complete set of invariants for F can be used as a set of complete invariants for E. Borel reducibility is also sometimes viewed as describing *Borel cardinality*, where the injection $\hat{f}$ is a Borel witness that the quotient of E injects into the quotient of F.

A Borel equivalence relation is said to be *countable* if all of its equivalence classes are countable. See [8] for an introduction to the theory of countable Borel equivalence relations. It is an important fact that there are maximal countable Borel equivalence relations under $\leq_B$, and such equivalence relations are said to be *universal* [5]. Our first theorem is that the resource bounded equivalence relations discussed above are universal (see Theorem 3.1).

Theorem 1.1. *Suppose $g(n) \geq n^2$ is a strictly increasing time-constructible function. Then any countable Borel equivalence relation E such that $\equiv_m^{\mathbf{TIME}(O(g^k))} \subseteq E \subseteq \equiv_T^{\mathbf{SPACE}(O(g^k))}$ is universal.*

For example, poly-time many-one equivalence and poly-time Turing equivalence are both universal countable Borel equivalence relations, corresponding to the case $g(n) = n^2$.

1.2 Ultrafilters, universality, and relativization

Ultrafilters on the σ-algebra of invariant Borel sets of a equivalence relation are important tools used in the study of Borel reducibility of equivalence relations. Such ultrafilters arise naturally from ergodic probability measures and generically ergodic topologies, and these types of ultrafilters have long been used in the subject. More recently, work in [11] and [10] has used ultrafilters related to Martin's ultrafilter on the Turing degrees to derive structural consequences about universal equivalence relations of various kinds. See [10] for an introduction to these ultrafilters. Here, we continue this investigation by discussing such ultrafilters for resource bounded equivalence relations from computability theory.

The ultrafilters we consider are defined via games very similar to the the game defining Martin's ultrafilter on the Turing invariant sets (see [12]). In these games, two players alternate defining the elements of an oracle x, and a set A of oracles is in the ultrafilter iff player I has a winning strategy in the game when A is the payoff set for player I. Martin has shown [12, 13] that for Borel Turing invariant sets A, we have that A is in this ultrafilter if and only if it contains a *Turing cone*, that is, a set of the form $\{x : x \geq_T y\}$

for some oracle y. It is folklore that Martin's proof generalizes to any quasi-order $\leq_Q$ such that given a strategy σ and an oracle x which is $\geq_Q$ an encoding of σ, then $x \equiv_Q \sigma * x$ where $\sigma * x$ is the outcome of playing σ against x. That is, Martin's game defines an ultrafilter on the $\equiv_Q$-invariant sets under AD so that a set is this ultrafilter if and only if it contains a $\leq_Q$-cone. While this condition encompasses a large number of quasi-orders, it does not include Turing equivalence restricted to sub-exponential time or space bounds, since strategies in games take an exponential amount of space to encode. Nor does it include many-one equivalence, or its time or space restricted versions. Here then, we may ask to what extent there can be an analogue of Martin's ultrafilter for these equivalence relations.

First, we note that standard relativization barriers show that there cannot be an ultrafilter defined via cones for these equivalence relations. For example, relativizing the theorem of Baker, Gill, and Solovay [2] giving oracles x and y relative to which $\mathbf{P}^x = \mathbf{NP}^x$ and $\mathbf{P}^y \neq \mathbf{NP}^y$ shows that both the oracles relative to which $\mathbf{P} = \mathbf{NP}$ and $\mathbf{P} \neq \mathbf{NP}$ are cofinal under poly-time Turing reducibility $\leq_T^{\mathbf{P}}$, and so there is no cone ultrafilter on the $\equiv_T^{\mathbf{P}}$-invariant sets. However, it turns out that Martin's game still defines an ultrafilter with interesting structure-preserving properties, even though it does not have a definition via cones.

Theorem 1.2. *Let E be a countable Borel equivalence relation as in Theorem 1.1. Then Martin's game in* [12] *defines an ultrafilter on the Borel E-invariant sets such that for any $A \in U$, $E \upharpoonright A$ is a universal countable Borel equivalence relation.*

There is some connection here with issues in computability and complexity theory surrounding the phenomenon of relativization. In recursion theory, proofs almost always relativize. That is, we can take most any proof in the subject, change all Turing machines used in the proof to grant them access to an additional oracle x, and the proof will remain valid. And while early sweeping attempts to formalize this phenomenon failed, such as Rogers' homogeneity conjecture (see [15, §13.1], [16, §12], [18], and [19]), Martin's ultrafilter on the Turing degrees yields a somewhat weaker explanation of the ubiquity of this phenomenon. If $\psi(\mathbf{x})$ is any reasonable statement about Turing degrees $\mathbf{x}$, then either $\psi(\mathbf{x})$ is true on a cone of oracles, or $\psi(\mathbf{x})$ is false on a cone of oracles. Hence, every reasonable fact about the Turing degrees eventually relativizes above some Turing cone.

As we've noted above,Baker, Gill, and Solovay's result implies that the poly-time analogue of this result is false. However, since Martin's cone theorem becomes true once we enlarge beyond iterated exponential time Turing equivalence, we can show that the connection between sets in our ultrafilter and cones (and hence the meta-theorem preventing the existence of relativization barriers on a cone) occurs once E grows beyond iterated exponential time Turing equivalence. That is, if E contains $\equiv_T^{\mathbf{ELEMENTARY}}$, then $A \in U$ if and only if A contains a $\leq_T^{\mathbf{ELEMENTARY}}$-cone. This fact tells us something interesting about relativization barriers themselves – that on a cone, such barriers can always be found inside the exponential hierarchy. Suppose ψ is a $\equiv_m^{\mathbf{P}}$ invariant property (such as whether two naturally defined relativized complexity classes coincide). Suppose also that there is a relativization barrier for ψ which itself relativizes, so that for every z, there are $x \geq_m^{\mathbf{P}} z$ and $y \geq_m^{\mathbf{P}} z$ relative to which $\psi(x)$ is true, and $\psi(y)$ is false. Then for a $\leq_m^{\mathbf{P}}$-cone of z, we can find such x and y so that $x, y \in \mathbf{ELEMENTARY}^z$. A consequence of this is that if ψ is invariant on $\equiv_T^{\mathbf{ELEMENTARY}}$-classes, then it must be either true or false on some $\leq_T^{\mathbf{ELEMENTARY}}$-cone, and hence ψ can not admit a Baker-Gill-Solovay-type relativization barrier.

In another direction, we show that the set of languages $x \in \mathcal{P}(2^{<\omega})$ for which $\mathbf{P}^x = \mathbf{NP}^x$ is in the ultrafilter U defined in Theorem 1.2. While oracles relative to which $\mathbf{P} = \mathbf{NP}$ are often considered to be somehow "rare" (for example meager [4] and Lebesgue null [3]) here we have a natural sense in which the set of oracles relative to which $\mathbf{P} = \mathbf{NP}$ is larger than its complement. Finally, we note that from the perspective of Borel cardinality, there are exactly as many oracles relative to which $\mathbf{P} = \mathbf{NP}$ as oracles to which $\mathbf{P} \neq \mathbf{NP}$; poly-time equivalence restricted to both sets is universal.

2 Preliminaries

We begin by reviewing some notation and conventions. Given any set S, we will often exploit the bijection via characteristic functions between its powerset $\mathcal{P}(S)$, and 2^S, the space of functions from S to $\{0, 1\}$, and move freely between these two representations. We use the notation 2^n for the set of finite binary strings of length n and $2^{\leq n}$ for finite binary strings of length $\leq n$. The set of all finite strings is noted $2^{<\omega}$. We use $r^\frown s$ to note the concatenation of the strings r and s.

Define $\mathcal{P}(2^{<\omega})$ to be the Polish space of subsets of $2^{<\omega}$. If $x, y \in \mathcal{P}(2^{<\omega})$, then their recursive join is $x \oplus y = \{0^\frown s : s \in x\} \cup \{1^\frown s : s \in y\}$. The recursive join of finitely many elements of $\mathcal{P}(2^{<\omega})$ is defined similarly. We use the notation A^{c} to denote the complement of a set A. A function $g \colon \mathbb{N} \to \mathbb{N}$ is said to be *time constructible* if $g(n)$ is computable in $O(g(n))$ time.

In computational complexity, the issue of correctly relativizing and modeling oracle access is a delicate and complicated matter. See [7] for a discussion of some of these issues. The theorems we will prove, however, will be quite robust with respect to this issue, and the only assumption we will make is the standard convention that oracle strings of length n may be queried only after time n.

If E and F are equivalence relations on the spaces X and Y, then a function $f\colon X \to Y$ is a *homomorphism from E to F* if for all $x, y \in X$ we have $x \; E \; y \to f(x) \; F \; f(y)$. A function $f\colon X \to Y$ is a *cohomomorphism from E to F* if for all $x, y \in X$ we have $f(x) \; F \; f(y) \to x \; E \; y$

3 A universality proof

In this section, we prove the following theorem from the introduction.

Theorem 3.1. *Suppose $g(n) \geq n^2$ is a strictly increasing time-constructible function. Then any countable Borel equivalence relation E such that $\equiv_m^{\mathbf{TIME}(O(g^k))} \subseteq E \subseteq \equiv_T^{\mathbf{SPACE}(O(g^k))}$ is universal.*

Proof. We let E_∞ note a universal countable Borel equivalence relation that is generated by a continuous action of $\mathbb{F}_2 = \langle \alpha, \beta \rangle$ on 2^ω. For example, we can use $E(\mathbb{F}_2, 2)$ for this purpose. We will show that E is universal by constructing a continuous embedding $\hat{f}$ of E_∞ into E.

The key to our proof is that given $\gamma \in \mathbb{F}_2$, we can code $\hat{f}(\gamma \cdot x)$ into $\hat{f}(x)$ rather sparsely so that if $|\gamma| = k$ is the length of γ as a reduced word, then strings of length n in $\hat{f}(w \cdot x)$ are coded by strings of length greater than $g^k(n)$ in $\hat{f}(x)$. From here our basic idea is as follows: given $x, y \in 2^\omega$ such that $x \not\!E_\infty y$ and a Turing reduction that runs in g^k time, we wait till we have finite initial segments of x and y witnessing that $\gamma \cdot x \neq y$ for any γ of length $\leq k$. Then if n is large enough, we can change the value of strings of length n in $\hat{f}(x)$ without changing $\hat{f}(y)$ restricted to strings of length $\leq g^k(n)$. This makes it easy to diagonalize. The remaining technical wrinkle of the proof is that we must be able to simultaneously do a lot of this sort of diagonalization.

We now give a precise definition of the coding we will use. Let $c : 2^{<\omega} \to 2^{<\omega}$ be the function where $c(r) = 0^{g(r)}{}^\frown 1^\frown r$ is $g(r)$ zeroes followed by a 1 followed by r. It is clear that if $x \in \mathcal{P}(2^{<\omega})$, then x can compute $c(x) = \{c(r) : r \in x\}$ in $O(g^k)$ time.

Given $f : 2^\omega \to \mathcal{P}(2^{<\omega})$, let $\hat{f} : 2^\omega \to \mathcal{P}(2^{<\omega})$ be defined by

$$\hat{f}(x) = f(x) \oplus c\left(\hat{f}(\alpha \cdot x) \oplus \hat{f}(\alpha^{-1} \cdot x) \oplus \hat{f}(\beta \cdot x) \oplus \hat{f}(\beta^{-1} \cdot x)\right).$$

While this definition of $\hat{f}$ is self-referential, it is not circular, and there is a unique function $\hat{f} \colon 2^\omega \to \mathcal{P}(2^{<\omega})$ with this property. This is because strings of length n in a language y will be coded by strings of much greater length in $f(x) \oplus c\,(y)$.

Now it is clear that given any f, the associated $\hat{f}$ is a homomorphism from E_∞ to E, since $\equiv_m^{\mathbf{TIME}(O(g^k))} \subseteq E$. We claim that if f is a sufficiently generic continuous function, then $\hat{f}$ will also be a cohomomorphism from E_∞ to $\equiv_T^{\mathbf{SPACE}(O(g^k))}$ and hence also a cohomomorphism from E_∞ to E. By generic, here, we mean for the following partial order for constructing a continuous (indeed, Lipschitz) functions from 2^ω to $\mathcal{P}(2^{<\omega})$. Our partial order $\mathbb{P}$ will consist of functions $p : 2^n \to \mathcal{P}(2^{\leq n})$ such that if $m < n$ and $r_1, r_2 \in 2^n$ extend $r \in 2^m$, then $p(r_1)$ and $p(r_2)$ agree on all strings of length $\leq m$. Given a $p : 2^n \to \mathcal{P}(2^{\leq n})$ and $r \in 2^n$, we will often think of $p(r)$ as a function from $2^{\leq n}$ to 2. If p has domain 2^n, then we will say p has height n. If $p, p^* \in \mathbb{P}$ are such that the height of p is m and the height of p^* is n, then say that p^* extends p, noted $p^* \leq_{\mathbb{P}} p$, if for all $r^* \in 2^n$ extending $r \in 2^m$, we have that $p^*(r^*)$ extends $p(r)$ (as functions).

If $p \in \mathbb{P}$, then we can define $\hat{p}$ analogously to the definition of $\hat{f}$ above. In particular, for each $\gamma \in \mathbb{F}_2$, we have some partial information about $p(\gamma \cdot r)$ based on the longest finite initial segment of $\gamma \cdot r$ that we know (recall that the action of $\mathbb{F}_2$ on 2^ω is continuous). Hence, given a finite string r, $\hat{p}$ will map r to a partial function from finite strings to 2 that amalgamates all this partial information.

Because of our coding scheme, if $r \in 2^{<\omega}$, the length of r is $|r| = n$, and $\gamma \in \mathbb{F}_2$ is of length k, then whether $r \in \hat{f}(\gamma \cdot x)$ is canonically coded into $\hat{f}(x)$ at some string of length greater than $g^k(n)$.

Let p_0 be the condition of height 1 where $p_0(r) = \emptyset$ for all r. Hence, if $f : 2^\omega \to \mathcal{P}(2^{<\omega})$ extends p_0, then every string in $f(x)$ must have length ≥ 2. For convenience, the generic function we construct will extend p_0. Now fix φ_e, which runs in g^k-time. Note that since $g(n) \geq n^2$, if f is $O(g^{k-1})$ for some k, then $f(n) \leq g(n)^k$ for sufficiently large n.

Suppose we are given a Turing reduction φ_e that runs in g^k time, and $r, s \in 2^m$ such that $\gamma \cdot r$ is incompatible with s for all $\gamma \in \mathbb{F}_2$ where $|\gamma| \leq k$. Let $D_{r,s,k,e}$ be the set of p of height $\geq m$ such that if p has height i, then there exists an $n \geq m$ so that if $t \in 2^i$, then $\hat{p}(t)$ is defined on all strings of length $\leq g^k(n)$, and for all $r^*, s^* \in 2^i$ extending r and s, we have that φ_e is not a g^k-time reduction of $\hat{p}(s^*)$ to $\hat{p}(r^*)$ as witnessed by some string of length n. We claim that $D_{r,s,k,e}$ is dense below p_0. The theorem will follow from this fact.

Given any $p \leq_{\mathbb{P}} p_0$ where p has height j, we must construct an extension p^* of p so that $p^* \in D_{r,s,k,e}$. Fix an n and an i such that $i \gg n \gg j$. Define q of height i where for all r, $q(r)$ contains only the strings in $p(r \upharpoonright i)$. Here we require $i \gg n$ so that for all $r \in 2^i$, $\hat{q}(r)$ is defined on all strings of length $\leq g^k(n)$. To determine how large n must be, we let n be variable for the next few paragraphs, while j and k are constant.

Given any $r^* \in 2^i$ we compute an upper bound on how many elements $\hat{q}(r^*)$ could have of length $\leq g^k(n)$. Clearly every $q(t)$ has less than 2^{j+1} elements. It will be enough to establish an upper bound on the number of words $\gamma \in \mathbb{F}_2$ so that some element of $q(\gamma \cdot r^*)$ is coded into $\hat{q}(r)$ via a string of length $\leq g^k(n)$.

Since q extends p_0, any element of any $q(\gamma \cdot r^*)$ must be a string of length ≥ 2. Now if $w \in \mathbb{F}_2$ is such that some string of length ≥ 2 in $q(\gamma \cdot r^*)$ is coded into $\hat{q}(r^*)$ below $g^k(n)$, then it must be that $g^{|\gamma|}(2) \leq g^k(n)$. Since $g(n) \geq n^2$, we have that $g^{|\gamma|}(2) \geq 2^{2^{|\gamma|}}$. Hence, the length of γ must be $\leq k + \log_2 \log_2(n)$. we see that there are most $O(\log_2(n))^2)$ such words γ Since there are $\leq 4^{l+1}$ words of $\mathbb{F}_2$ of length $\leq l$.

Since each $q(\gamma \cdot r^*)$ must have less than 2^{j+1} elements, $\hat{q}(r^*)$ contains at most $O((\log_2(n))^2)$ strings of length $\leq g^k(n)$. Let S be this set of all possible strings in $\hat{q}(r^*)$ and let n be large enough so that $|S| < n$. Note that S does not depend on r^*.

We see now that amongst all of the r^* extending r, there are $\leq 2^{n-1}$ possibilities for what any $\hat{q}(r^*)$ could be: they are all elements of $\mathcal{P}(S)$. Let $u_0, u_1, \ldots$ be a listing of the elements of $\mathcal{P}(S)$. Recall that based on our definition of recursive join and $\hat{p}$, strings of length $n-1$ in $p(s^*)$ are coded into $\hat{p}(s^*)$ using strings of length n that begin with 0. Define p^* to be equal to q except on extensions s^* of s. There, if σ is the lth string of length $n-1$, then put $\sigma \in p(s^*)$ if and only if σ is not accepted by φ_e run relative to u_l. $\square$

4 Relativization barriers and largeness notions for sets of poly-time degrees

Martin's ultrafilter is defined by a game where two players alternate defining the bits of a real, as follows:

Definition 4.1. Let U be the collection of $\equiv^{\mathbf{P}}_m$-invariant Borel sets $A \subseteq \mathcal{P}(2^{<\omega})$ such that player I has a winning strategy in the following game G_A. Players I and II alternate defining which strings are in a language x where on turn n, player I decides membership in x for all strings of length n beginning with a 0, and on turn n, player II decides membership in x for all strings of length n beginning with a 1, so x is the recursive join of the languages played by I and II. Then player I wins the game if $x \in A$.

We now show that U is a σ-complete ultrafilter for $\equiv_m^{\mathbf{P}}$-invariant sets.

Proposition 4.2. *U is a σ-complete ultrafilter on the $\equiv_m^P$-invariant Borel sets.*

Proof. Because A is $\equiv_m^{\mathbb{P}}$-invariant, it is easy to check that for any definition of a recursive join $\oplus$ such that the map $(x, y) \mapsto x \oplus y$ is poly-time many-one equivalent to our canonical definition in Section 2, the winner of the game does not change. Hence, we can see that $A \in U$ iff $A^{\mathrm{c}} \notin U$ by simply switching the roles of players I and II in G_A. Similarly, given countably many strategies $\sigma_0, \sigma_1, \ldots$ for player I in G_A, we can produce a real that is simultaneously a winning outcome of $\sigma_0, \sigma_1, \ldots$, being the recursive join of the plays of player I according to each of these strategies. Hence, U is σ-additive. $\square$

Next, an easy extension of Theorem 3.1 shows the following:

Theorem 4.3. *Let E be a countable Borel equivalence relation as in Theorem 3.1. Then Definition 4.1 defines an ultrafilter on the Borel E-invariant sets such that for any $A \in U$, $E \restriction A$ is a universal countable Borel equivalence relation.*

Proof. Given a strategy σ for player I in the game G_A, and any $y \in \mathcal{P}(2^{<\omega})$, let $\sigma(y)$ be the outcome of playing the strategy σ against player II playing y. Then if we replace the definition of $\hat{f}$ in the proof with

$$\hat{f}(x) = \sigma\left(f(x) \oplus c\left(\hat{f}(\alpha \cdot x) \oplus \hat{f}(\alpha^{-1} \cdot x) \oplus \hat{f}(\beta \cdot x) \oplus \hat{f}(\beta^{-1} \cdot x)\right)\right)$$

then $\hat{f}$ remains an embedding of E_∞ into E. Furthermore, $\hat{f}$ must then be an embedding whose range is contained in the set A, since σ is a winning strategy for player I in G_A. $\square$

Now an easy argument shows that if $A = \{x : \mathbf{P}^x = \mathbf{NP}^x\}$, then there is a winning strategy for player I in the game G_A; player I simply ensures that the outcome of the game is a language that is poly-time equivalent to its **PSPACE** completion. Thus, we have the following corollary, since the original proof of Theorem 3.1 has a generic enough range to run the argument of [4] to show that $\mathrm{ran}(\hat{f}) \subseteq A^{\mathrm{c}}$.

Corollary 4.4. *Let $A = \{x : \mathbf{P}^x = \mathbf{NP}^x\}$. Then $\equiv_m^{\mathbf{P}} \restriction A$, $\equiv_m^{\mathbf{P}} \restriction A^c$, $\equiv_T^{\mathbf{P}} \restriction A$, and $\equiv_T^{\mathbf{P}} \restriction A^c$, are all universal countable Borel equivalence relations.*

We finish by noting that the ultrafilter U in Definition 4.1 can be combined with Theorem 4.3 to provide an alternate way of proving many of the the results in section 3 of [11]; one simply uses them to replaces Martin's measure and arithmetic equivalence in that paper.

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